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Twisted algebra R-matrices and S-matrices for $b_n^{(1)}$ affine Toda solitons and their bound states

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ABSTRACT

We construct new $U_q(a_{2n-1}^{(2)})$ and $U_q(e_6^{(2)})$ invariant R-matrices and comment on the general construction of R-matrices for twisted algebras. We use the former to construct S-matrices for $b_n^{(1)}$ affine Toda solitons and their bound states, identifying the lowest breathers with the $b_n^{(1)}$ particles.

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1 Introduction

Affine Toda field theories (ATFTs) based on affine algebras $\hat{g}$ have quantized $\hat{g}^\vee$ charge algebras[1] [2], and consequently S-matrices for affine Toda solitons can be constructed from $U_q(\hat{g}^\vee)$ R-matrices, as has been done for $a_n^{(1)}$ by Hollowood [22]. For more details we refer the reader to [3, 17].

For cases where $\hat{g}$ is untwisted and nonsimply-laced, the least understood of the ATFTs, twisted algebra R-matrices must be used. It has been a longstanding problem, recently solved by Delius, Gould and Zhang [1], to construct R-matrices for quantized twisted affine algebras. In this paper we build on their results to produce new R-matrices associated with the quantum deformations of the algebras $a_{2n-1}^{(2)}$ and $e_6^{(2)}$.

In particular our $a_{2n-1}^{(2)}$ R-matrices allow us to construct S-matrices for $b_n^{(1)}$ solitons and their bound states. Our main result here is that the lowest scalar bound states ('breathers') may be identified with the $b_n^{(1)}$ ATFT quantum particles.

The layout of the paper is as follows: in section 2 we give our conventions for the quantized universal enveloping algebras. In section 3 we summarize the previous constructions of R-matrices for untwisted and twisted algebras and in section 4 give our new constructions. We finish this section with some general comments on the construction of R-matrices. After some more general material in section 5, in section 6 we give details of the $b_n^{(1)}$ soliton and breather S-matrices constructed using the $a_{2n-1}^{(2)}$ R-matrices found in section 4, and we finish this section with some comments on the status of imaginary coupling affine Toda field theories in general.

We follow the conventions of [3], in which other details such as Coxeter and dual Coxeter numbers may be found.

2 Quantized enveloping algebras

For ease of notation we shall everywhere write ${}_q\mathcal{A}$ for $U_q(\mathcal{A})$.

The generators of ${}_qg$ are e_i, f_i, h_i, $i = 1, .., \mathrm{rank}(g)$, satisfying the q–deformed Serre relations

[1] The dual $\mathcal{A}^\vee$ of an algebra $\mathcal{A}$ is obtained by replacing its simple roots by co-roots, equivalent to reversing the arrows on its Dynkin diagram.

(see [1] for these and more details) and

$$[h_i, e_j] = (\alpha_i, \alpha_j)e_j \ , \ \ [h_i, f_j] = -(\alpha_i, \alpha_j)f_j \ , \ \ [h_i, h_j] = 0 \ , \tag{2.1}$$

$$[e_i, f_j] = \delta_{ij}\frac{q^{h_i} - q^{-h_i}}{q_i - q_i^{-1}} \ , \ \ q_i = q^{\frac{1}{2}(\alpha_i,\alpha_i)} \ , \tag{2.2}$$

with coproduct[2]

$$\begin{aligned} \Delta(e_i) = q^{-h_i/2} \otimes e_i + e_i \otimes q^{h_i/2} \ , \qquad \Delta(f_i) = q^{-h_i/2} \otimes f_i + f_i \otimes q^{h_i/2} \ , \\ \Delta(q^{\pm h_i/2}) \ = \ q^{\pm h_i/2} \otimes q^{\pm h_i/2} \ . \end{aligned} \tag{2.3}$$

The generators of ${}_q g^{(k)}$ for an affine algebra also satisfy equations (2.1,2.2,2.3), but where the index i now runs from 0. Since the roots appear explicitly in (2.1), we have to choose a normalization, which is that the long roots of a finite algebra g satisfy $|\alpha_{long}|^2 = 2$, whereas for those of an affine algebra $g^{(k)}$ satisfy $|\alpha_{long}|^2 = 2k$.

We shall also need Reshetikhin's result [4] on the spectral decomposition of the ${}_q g$ R-matrix. This holds in any tensor product of two irreducible representations $V_\lambda \otimes V_\mu = \oplus_\nu V_\nu$ for which each irreducible representation V_ν occurs with multiplicity at most one, in which case we have

$$R_{\lambda\mu} = \sum_\nu \epsilon(\nu) q^{C_2(\nu)-C_2(\lambda)-C_2(\mu)} \ , \tag{2.4}$$

where $\epsilon(\nu) = \pm 1$, and where $C_2(\lambda)$ is the Casimir operator normalized so that $C_2(\text{adjoint}) = h^\vee$, the dual Coxeter number of g.

3 R-matrices for untwisted algebras

The construction of Delius et al. [1] is an extension of the Tensor Product Graph method [11, 10, 12]. We outline this method in the case of untwisted algebras now.

Let $g^{(1)}$ be an untwisted affine algebra. It is clear from the defining generators and relations that the generators e_i, f_i, h_i with $i = 1, .., \text{rank}(g)$ generate a ${}_q g$ subalgebra of ${}_q g^{(1)}$.

The physical particles transform as finite-dimensional ${}_q g^{(1)}$ representations, and so the R-matrices of interest are those in the tensor product $V_a \otimes V_b$ of finite-dimensional ${}_q g^{(1)}$ modules.

[2]Note that this is the opposite coproduct to that used in [2], so that the q in that paper is the inverse of q here

Given a representation π_a of ${}_qg^{(1)}$ on a vector space V_a, we can then find another representation $\pi_a(x)$ by taking

$$\pi_a(x)(e_0) = x\pi_a(e_0)\ ,\ \ \pi_a(x)(f_0) = x^{-1}\pi_a(f_0)\ ,\ \ \pi_a(x)(h_0) = \pi_a(h_0)\ ,$$
$$\pi_a(x)({}_qg) = \pi_a({}_qg)\ , \tag{3.1}$$

which is simply a gauge transformation. x is the spectral parameter.

We take the $\check{R}_{a,b}(x)$ to satisfy

$$\check{R}_{a,b}(x)(\pi_a(x)\otimes\pi_b(1))\,(\Delta({}_qg)) = (\pi_b(1)\otimes\pi_a(x))\,(\Delta({}_qg))\,\check{R}_{a,b}(x)\ , \tag{3.2}$$

$$\check{R}_{a,b}(x)\check{R}_{ba}(x^{-1}) = \mathbf{1}\ ,\ \ \check{R}_{a,b}(0) = \check{R}_{a,b}\ , \tag{3.3}$$

where $\check{R}_{a,b}$ is the R-matrix of ${}_qg$. This fixes $\check{R}_{a,b}(x)$ up to an overall factor $f(x)$ such that $f(x)f(x^{-1}) = 1, f(0) = 1$. Since $\mathbf{P}\,\check{R}_{a,b}$ commutes with the action of ${}_qg$ (where $\mathbf{P}$ is the permutation matrix), $\mathbf{P}\check{R}_{a,b}(x)$ must be a sum of projectors onto the irreducible ${}_qg$-components of $V_a\otimes V_b$. We denote the ${}_qg$–projector onto representation V_c by $\mathbf{P}\check{P}_c$, so that

$$\check{R}_{a,b}(x) = \sum_{c_i} f_i(x)\check{P}_{c_i}\ . \tag{3.4}$$

If each irreducible ${}_qg$-representation V_c occurs no more than once in $V_a\otimes V_b = \sum_i V_{c_i}$, then we say that it is multiplicity-free. It is only in this case that we can determine $\check{R}_{a,b}$ by the Tensor Product Graph method alone. An immediate requirement for the construction of $R_{a,a}$ is that V_a is itself irreducible with respect to ${}_qg$. If $V_a\otimes V_b$ is multiplicity free then we find[3]

$$\frac{f_j(x)}{f_i(x)} = \Big\langle C_2(V_{c_i}) - C_2(V_{c_j})\Big\rangle_{\epsilon(c_i)\epsilon(c_j)}\ , \tag{3.5}$$

for each pair of representations V_{c_i} and V_{c_j} such that $V_{c_j}\subset V_{c_i}\otimes\tilde{V}$, where $\tilde{V}$ is the ${}_qg$ representation corresponding to the extra generator e_0, and where

$$\langle a\rangle_\pm = \frac{1\pm xq^a}{x\pm q^a}\ . \tag{3.6}$$

The R-matrix is then determined up to an overall scalar factor, which is usually fixed by taking one of the modules to have coefficient 1.

A standard way of presenting this information is the Tensor Product Graph itself (or TPG for short), in which the components V_{c_i} and V_{c_j} are connected by an arrow from i to j if and only if

[3] n.b. this is different from the results in [1] as we have chosen to normalize C_2 differently here.

$V_{c_j} \subset V_{c_i} \otimes \tilde{V}$, and the arrow is labelled by a. (The TPG edge is thus directed, and in the case that $\tilde{V}$ is self-conjugate, as will always be true for us, an edge with label a is equivalent to the reverse-directed edge with label $-a$). We choose the module with coefficient 1 to be that with the highest weight, and write the TPG with arrows directed only away from it. In [12] it was argued that in the untwisted case the product $\epsilon(c_i)\epsilon(c_j)$ for two components joined by a TPG edge is always -1.

3.1 R–matrices for twisted algebras

A twisted quantum affine Lie algebra ${}_q g^{(k)}$ has generators e_i, f_i, h_i for $i = 0, .., l$ also satisfying equations (2.1), (2.2), (2.3), but where α_i are now the simple roots of a twisted affine algebra.

Exactly as in the case of an untwisted quantum affine Lie algebra, there are subalgebras of ${}_q g^{(k)}$ generated by $e_{i'}, f_{i'}, h_{i'}$ where $\{i'\}$ are any subset of $\{0, .., l\}$. Such a subalgebra is a direct sum

$$\oplus \; (\,{}_{q_i} h_i \,) \;, \tag{3.7}$$

where the Dynkin diagrams of h_i are the different components of the subdiagram of that of $g^{(k)}$ consisting of the nodes $\{i'\}$. There is one subtlety involved[4], which is that the roots $\{\alpha_{i'}\}$ of any particular subalgebra ${}_{q_i} h_i \subset {}_q g^{(k)}$ may not agree with the specific normalization for the root lengths we have chosen. If $2\imath$ is the square of the longest root in the set $\{\alpha_{i'}\}$, then we have

$$q_i = q^{\imath} \;. \tag{3.8}$$

This can be different for the different components h_i in $g^{(k)}$.

In particular, as noted by Delius et al. in [1], one can always obtain g_0 as a subalgebra of $g^{(k)}$, where g_0 is the subalgebra of g invariant under the order k automorphism defining the twist,

$${}_{q^{\imath}} g_0 \subset {}_q g^{(k)} \;. \tag{3.9}$$

The TPG method now carries on as before, but the representation $\tilde{V}$ in which the extra generator of the twisted quantum affine algebra transforms is no longer the adjoint representation of ${}_{q^{\imath}} g_0$. If a ${}_q g_0^{(1)}$ R-matrix exists for the same g_0-modules its TPG will therefore be topologically different: the two TPGs will correspond to different 'Baxterizations' [13] of the centralizer algebra (the Birman-Wenzl-Murakami algebra in the b-, c- and d_n cases).

[4]This was passed over in [1]

The crux of the argument of Delius et al. [1], was to establish the relative parities of the components of the TPG in this case. (Note that in their notation Lie algebras are denoted by L in contrast to our g.) They observed that, when the ${}_q g^{(k)}$ representations are irreducible ${}_{q^\imath} g_0$ representations, the relative parity of two linked components V_{c_i} and V_{c_j} in the TPG is $+1$ if they are part of the same ${}_q g$ representations, and -1 if they are parts of different ${}_q g$ representations. Following this observation, the rest of the argument is identical to that in the case of untwisted algebras, with the understanding that the 'block' in equation (3.5) should be replaced by

$$\frac{f_j(x)}{f_i(x)} = \Big\langle\ \imath\ \Big(\ C_2(V_{c_i}) - C_2(V_{c_j})\ \Big)\ \Big\rangle_{\pm}\ , \tag{3.10}$$

for each pair of representations V_{c_i} and V_{c_j} such that $V_{c_j} \subset V_{c_i} \otimes \tilde{V}$, where $\tilde{V}$ is the ${}_{q^\imath} g_0$ representation corresponding to the extra generator e_0.

Our extension of the work of [1] is based on the observation that there may be more than one choice of finite dimensional subalgebra h of $g^{(k)}$ such that $\text{rank}(h) = \text{rank}(g_0)$: such an h is obtained by deleting one node from the Dynkin diagram of $g^{(k)}$. In this paper we only consider the cases where the h we obtain in this way is simple; we give these, along with the index $\imath$ in each case, in the table 1 below: g_0 is the choice given in [1], and g_0' is the only simple alternative. We recall that ${}_q g^{(1)}$-modules may be reducible as ${}_q g$-modules[5], and that, when the R-matrices are used to construct S-matrices, it is the fundamental ${}_q g^{(1)}$-modules (those whose highest component is a fundamental ${}_q g$-module) which correspond to the physical particle multiplets. We observe that the twisted algebra multiplets, the ${}_q g^{(k)}$-modules, are always a subset of the ${}_q g^{(1)}$-modules, and in table 1 we give the decomposition of these as ${}_q g$-, ${}_{q^\imath} g_0$- and ${}_{q^\imath} g_0'$-modules, where V_μ is the module with highest weight μ, and the λ_i are the fundamental weights. For ease of notation we also write $V_i \equiv V_{\lambda_i}$; V_0 is the singlet.

4 $R-$matrices from g_0'

We now consider, for each of the twisted algebras in turn, whether using g_0' instead of g_0 enables us to construct new $R-$matrices. As Delius et al. point out, we can only hope to construct R-matrices for those ${}_q g^{(k)}$ multiplets which are ${}_{q^\imath} h$-irreducible.

[5] For a full listing of those currently known, see [7].

4.1 $a_{2n-1}^{(2)}$

We see that this algebra has a ${}_q d_n$ as well as a ${}_{q^2} c_n$ subalgebra and that the multiplets are (mostly) irreducible modules of the former. Hence we can now construct the TPG for $R_{a,b}$ if $a+b \le n$, as in this case $V_a \otimes V_b$ is multiplicity-free. We find that the TPG is of the now-familiar form first seen in [10],

$$\begin{array}{ccccccc}
\mu_a+\mu_b & \to & \mu_{a+1}+\mu_{b-1} & \cdots\ \cdots & \to & \mu_{a+b-1}+\mu_1 & \to & \mu_{a+b} \\
\downarrow & & \downarrow & & & \downarrow & & \\
\mu_{a-1}+\mu_{b-1} & \to & \mu_a+\mu_{b-2} & \cdots\ \cdots & \to & \mu_{a+b-2} & & \\
\vdots & & \vdots & & & & & \\
\downarrow & & \downarrow & & & & & \\
\mu_{a-b+1}+\mu_1 & \to & \mu_{a-b+2} & & & & & \\
\downarrow & & & & & & & \\
\mu_{a-b} & & & & & & &
\end{array} \tag{4.1}$$

where the $\mu_i = \lambda_i$ (for $i = 1, .., n-2$), $\mu_{n-1} = \lambda_{n-1} + \lambda_n$, and where the node μ denotes the irreducible representation V_μ in all cases except μ_n, which denotes $V_{2\lambda_{n-1}} \oplus V_{2\lambda_n}$. The columns of the graph alternate in parity, with the first column having positive parity. We obtain

$$\check{R}_{a,b}^{(TPG)}(x) = \sum_{p=0}^{b} \sum_{r=0}^{b-p} \prod_{i=1}^{p} \langle a-b+2i \rangle_- \prod_{j=1}^{r} \langle 2n-a-b+2j \rangle_+ \check{P}_{\mu_{a+p-r}+\mu_{b-p-r}} \, . \tag{4.2}$$

As an example, consider $\check{R}_{1,1}$, which can be constructed using either the ${}_{q^2} c_n$ or the ${}_q d_n$ invariance. The relevant TPGs are given in the second row of table 2. Remembering that for c_n,

$$\dim V_{2\lambda_1} = n(2n+1) \ , \ \ \dim V_2 = (n-1)(2n+1) \ , \ \ \dim V_0 = 1 \ , \tag{4.3}$$

and that for d_n,

$$\dim V_{2\lambda_1} = (n+1)(2n-1) \ , \ \ \dim V_2 = n(2n-1) \ , \ \ \dim V_0 = 1 \ , \tag{4.4}$$

we see that the two R-matrices have the same rank for all x. Since these are both R-matrices for ${}_q a_{2n-1}^{(2)}$, they are in fact similar, being related by a gauge transformation.

Now examine the first row of table 2. The ${}_{q^2}c_n^{(1)}$ R–matrix is also ${}_{q^2}c_n$ invariant, but has a different TPG from the ${}_{q^2}c_n \subset {}_q a_{2n-1}^{(2)}$ R–matrix. Comparing these two R-matrices, we see that they give the same braid operator in the $x \to 0$ limit: as noted, they correspond to two different 'Baxterizations' of this braid operator.

The TPGs of the ${}_{q^2}c_n \subset {}_{q^2}c_n^{(1)}$ (top left in table 2) and ${}_q d_n \subset {}_q a_{2n-1}^{(2)}$ (bottom right) R-matrices are superficially similar, but of course the parities are different and the nodes of the former are ${}_{q^2}c_n$, not ${}_q d_n$, modules. We see, however, that if we replace q^{2n+2} by $-q^{2n}$ in the x-dependent coefficients of the former, we obtain those of the latter. Furthermore, making this change in the ${}_{q^2}c_n$ BWM algebra [14] maps it (and thus the projectors onto the various representations) to the ${}_q d_n$ BWM algebra. Combining these, we find that applying $q^{2n+2} \mapsto -q^{2n}$ to the full R-matrix maps the one into the other. We expect this to apply to the higher R-matrices too, a fact which may have significance for affine Toda solitons.

Similarly, applying $q^{2n-2} \mapsto -q^{2n}$ to the ${}_q d_n \subset {}_q d_n^{(1)}$ R-matrix gives us precisely the ${}_{q^2}c_n \subset {}_q a_{2n-1}^{(2)}$ R-matrix; Jimbo obtains his ${}_q a_{2n-1}^{(2)}$ R-matrix [5] in this way. This is in contradiction to the note in [15], which suggested that the arrows in table 2 should be vertical, with $q^{2n+2} \mapsto -q^{2n}$ mapping the ${}_{q^2}c_n \subset {}_{q^2}c_n^{(1)}$ to the ${}_{q^2}c_n \subset {}_q a_{2n-1}^{(2)}$ R-matrix, and so on.

4.2 $a_{2n}^{(2)}$

For ${}_q a_{2n}^{(2)}$ Delius et al. were able to find all the R-matrices since the relevant ${}_q a_{2n}$ representations are all irreducible with $g_0 = b_n$ (so that $g_0 = so(N+1)$ for $g^{(k)} = a_N^{(2)}$). They found R-matrices which correspond to the TPG (4.1) above for *all* a, b, with $\mu_i = \lambda_i$ for $i = 1, .., n-1$, $\mu_n = 2\lambda_n$ and $\mu_i = \mu_{2n+1-i}$ for $i = n+1, .., 2n$. These R-matrices thus include the $a, b \to 2n+1-a-b$ fusion which does not occur for non-self-dual algebras.

4.3 $d_{n+1}^{(2)}$

There is only one possible simple Lie algebra Dynkin diagram obtainable by deleting one node from the $d_{n+1}^{(2)}$ diagram, and so we cannot add to the results of [1] here, except to note that with our normalisations, ${}_{q^2}b_n \subset {}_q d_{n+1}^{(2)}$, rather than ${}_q b_n \subset {}_q d_{n+1}^{(2)}$ as in [1].

4.4 $e_6^{(2)}$

From table 1 we see that the 27 dimensional representation of ${}_qL = {}_qe_6$ is an irreducible representation of ${}_{q^\iota}q_0' = {}_{q^2}c_4$, and since the tensor product of two 27 dimensional ${}_{q^2}c_4$ representations is multiplicity free, we can construct the TPG for $R_{\underline{27},\underline{27}}$,

$$\begin{array}{ccccc} V_{2\lambda_2} & \rightarrow^1 & V_{\lambda_1+\lambda_2} & \rightarrow^4 & V_2 \\ \downarrow_3 & & \downarrow_3 & & \\ V_4 & \rightarrow^1 & V_{2\lambda_1} & & \\ \downarrow_6 & & & & \\ V_0 & & & & \end{array}$$

where the nodes denote irreducible ${}_{q^2}c_4$ representations. Again the columns of the graph alternate in parity, with the first column having positive parity.

4.5 $d_4^{(3)}$

In this case we find that the 8 dimensional representation of ${}_qd_4$, which was the reducible $\underline{7} \oplus \underline{1}$ representation of ${}_{q^\iota}g_0 = {}_{q^3}g_2$, is the irreducible $\underline{8}$ of ${}_{q^\iota}g_0' = {}_qa_2$. Unfortunately, since

$$\underline{8} \otimes \underline{8} = \underline{1} \oplus 2(\,\underline{8}\,) \oplus \underline{10} \oplus \overline{\underline{10}} \oplus \underline{27} \tag{4.5}$$

contains two copies of $\underline{8}$, we are not able to construct $R_{\underline{8},\underline{8}}$.

4.6 Comments

We have already observed that each ${}_qg^{(k)}$ multiplet is also a ${}_qg^{(1)}$ multiplet, and of course its decomposition does not depend on whether we choose g_0 or g_0'. We also observe (as we saw in the example of R_{11} for ${}_qa_{2n-1}^{(2)}$ above) that the R-matrices obtained by using g_0 and g_0' are similar.

In particular, the R-matrices obtained by using g_0 and g_0' have the same rank for all values of x and x' respectively, where x and x' are the spectral parameters in the two versions of the R-matrix and are related by $x^{n_0} = x^{n_0'}$, where the n_0, n_0' are the Kac marks of the extra root in the two cases, i.e. $x = x'$ except for $g^{(k)} = a_{2n}^{(2)}$.

Heuristically this is most easily seen by recalling that, in theories with a ${}_q\hat{g}$ charge algebra, all the charges corresponding to step operators transform non-trivially under a Lorentz boost: the algebra is in the 'spin' gradation, equal for simply-laced algebras to the principal gradation. The similarity transformation to the homogeneous gradation R-matrix then shifts all the rapidity (spectral parameter) dependence to the generator corresponding to the extending root. This may be chosen in different ways, but these will always be similar, and the rank of the direct channel process cannot depend on this choice.

We also observe that singularities of the ${}_qg^{(k)}$ R-matrix are also singularities of the ${}_qg^{(1)}$ R-matrix. For example, compare the ${}_qe_6^{(2)}$ R-matrix above with the ${}_qe_6^{(1)}$ R-matrix for the $\underline{27}$ of ${}_qe_6$ given by the TPG

$$V_{2\lambda_1} \ \to^1 \ V_2 \ \to^2 \ V_1 \ .$$

As we see, however, the reverse implication is not true: the ${}_qg^{(k)}$ R-matrix has further singularities, all on links between modules with the same parity. This, pointed out in [1] as the fact that linked ${}_{q^\imath}g_0$ modules have the same parity if they belong to the same ${}_{q^\imath}g$-module, and opposite parities if they belong to different ${}_{q^\imath}g$-modules, allows us to observe that the rational limit ($q \to 1$ with $x = q^u$) of a ${}_qg^{(k)}$ R-matrix is the corresponding $Y(g)$ (Yangian) R-matrix, for all k, and not merely, as is well-known, for $k = 1$.

A further observation is that

$$g_0^{\vee(1)\vee} = g^{(k)} \ , \tag{4.6}$$

for some $k > 1$. (For example, consider $g_0 = c_n$ and $g = a_{2n-1}$.) The significance of this will be drawn out when we come to discuss affine Toda solitons.

Finally we note that we have not considered all possible ${}_{q^\imath}h \subset {}_qg^{(k)}$. For $k = 1$, any finite dimensional Lie algebra which is a subalgebra of $g^{(1)}$ is also a subalgebra of g, so that we do not expect to be able to produce any new R-matrices by choosing different h; we expect that any ${}_qg^{(1)}$ representations which is irreducible with respect to ${}_{q^\imath}h$ will also be irreducible with respect to ${}_qg$. We have also not considered here any non-simple ${}_{q^\imath}h \subset {}_qg^{(k)}$, e.g. those obtained by deleting an interior node of the $g^{(k)}$ Dynkin diagram.

5 S-matrices for affine Toda solitons and the identification of x and q.

We recall from [2] that the $\hat{g}$ affine Toda field theories have ${}_q\hat{g}^\vee$ charge algebras, that the quantum solitons are expected to transform as irreducible ${}_q\hat{g}^\vee$ multiplets, and that the S-matrix for the scattering of solitons in irreducible representations a, b is expected to be of the form

$$S_{a,b}(\theta) = S^{(0)}_{a,b}(\theta)\ \tau_{21}\ \check{R}^{(TPG)}_{a,b}(x_{a,b}(\theta), q)\ \tau_{12}^{-1}\ , \tag{5.1}$$

where $S^{(0)}$ is a scalar factor, τ denotes the transformation from the homogeneous to the spin gradation, and the value of $x_{a,b}(\theta)$ depends on the particular way we choose to construct the TPG.

From the explicit construction of the charge algebra given in [2], we can find the θ dependence of $x_{a,b}(\theta)$, as detailed in [3]. Provided we choose an extending root e_0 for which the Kac mark is 1, we find

$$x_{a,b}(\theta) = \xi_{a,b}\ \exp\left(\left[\frac{4\pi i}{\beta^2}h - h^\vee\right]\theta\right)\ , \tag{5.2}$$

where $\xi_{a,b}$ is an overall constant which can be fixed by demanding crossing symmetry. In many cases $\xi_{a,b}$ can also be fixed by demanding crossing symmetry in $S_{1,1}$, and then constructing the other S-matrices by the fusion procedure.

It is also possible to deduce that

$$q^{\alpha_i^\vee\cdot\alpha_j^\vee} = \exp\left(\frac{4\pi^2 i}{\beta^2}\,(\alpha_i^\vee\cdot\alpha_j^\vee)\right)\ , \tag{5.3}$$

for all pairs of dual simple roots $\alpha_i^\vee, \alpha_j^\vee$. Given our standard root normalizations, it is possible to find a pair $\alpha_i^\vee, \alpha_j^\vee$ whose inner product is (-1) for all $\hat{g}$ except $\hat{g} = c_n^{(1)}$, and as a result we find

$$q = \begin{cases} \exp(\frac{4\pi^2 i}{\beta^2}) & , \quad \hat{g} \neq c_n^{(1)}, \\ \pm\exp(\frac{4\pi^2 i}{\beta^2}) & , \quad \hat{g} = c_n^{(1)}. \end{cases} \tag{5.4}$$

Recall now the ansatz used in [3],

$$\tilde{x}(\theta) = e^{h\lambda\theta}\ , \qquad \tilde{q} = e^{\omega i\pi}\ , \tag{5.5}$$

with

$$\lambda = \frac{4\pi}{\beta^2} - \frac{h^\vee}{h}\ , \qquad \omega = \frac{h}{h'}\left(\frac{4\pi}{\beta^2} - t\right)\ . \tag{5.6}$$

This gave the correct masses for direct channel soliton poles at principal roots of $\tilde{x} = \tilde{q}^r$ and crossed channel poles at $\tilde{x} = \tilde{x}(i\pi)\tilde{q}^{-r}$, but did not always agree with crossing symmetry of the R-matrices. This is because in some cases ($a_n^{(2)}$ and $c_n^{(1)}$ ATFTs) the correct R-matrix to use in (5.1), $\check{R}_{a,b}(x_{a,b}(\theta), q)$, is not equal to $\check{R}_{a,b}(\tilde{x}(\theta), \tilde{q})$.

The final result is that the expected soliton masses and crossing symmetry are all obtained perfectly for S-matrices corresponding to the R-matrices in this paper and in [1]. There is thus no longer any barrier to the construction of soliton S-matrices corresponding to known R-matrices (all those for $a_{2n}^{(2)}$ ATFTs and S_{nn} in $c_n^{(1)}$ ATFTs), and we expect to deal with these in a future paper. For the moment, we limit ourselves to the construction of S-matrices for the $b_n^{(1)}$ affine Toda solitons, the untwisted nonsimply-laced case for which we have the most R-matrix information.

6 S-matrices for $b_n^{(1)}$ affine Toda solitons and their bound states

6.1 Soliton S-matrix

For details of affine Toda field theories we refer the reader to our earlier paper [3] and to a recent review [17]. To construct the S-matrices for $b_n^{(1)}$ solitons we recall that the $b_n^{(1)}$ ATFT has a ${}_q a_{2n-1}^{(2)}$ charge algebra and thus uses the R-matrices constructed in sub-section 4.1. For clarity we give both $\tilde{x}$, $\tilde{q}$ and $x_{a,b}$, q :

$$\tilde{x} = (-1)^{a+b} x_{a,b} = e^{2n\lambda\theta} , \qquad \text{in which} \quad \lambda = \frac{4\pi}{\beta^2} - \frac{2n-1}{2n} ,$$

and

$$\tilde{q} = -q = e^{\omega i\pi} , \qquad \text{in which} \quad \omega = \frac{4\pi}{\beta^2} - 1 , \tag{6.1}$$

and for later convenience we also introduce

$$\mu \equiv -i\frac{n\lambda}{\pi}\theta . \tag{6.2}$$

We conjecture the exact S-matrix for the scattering of two elementary solitons in $b_n^{(1)}$ ATFT for $a + b \leq n$ to be

$$S_{a,b}(\theta) = F_{a,b}(\mu(\theta)) k_{a,b}(\theta)\, \tau_{21} \check{R}_{a,b}^{(TPG)}(x_{a,b}, q) \tau_{12}^{-1} \tag{6.3}$$

in which $\check{R}_{a,b}^{(TPG)}(x_{a,b}, q)$ is given by (4.2) (and is in this case equal to $\check{R}_{a,b}^{(TPG)}(\tilde{x}, \tilde{q})$, because the TPG labels are all odd/even precisely as $a + b$ is odd/even[6]), $k_{a,b}$ is the R-matrix fusion factor

[6]True also for the $d_{n+1}^{(2)}$ soliton S-matrices in [3].

and is the same as in the $d_{n+1}^{(2)}$ case, (5) of [3], and the overall scalar factor $F_{a,b}$ will be determined below. The S-matrix acts as an intertwiner on the modules of the ${}_q d_n$-representations defined in subsection 4.1:

$$S_{a,b}(\theta) : V_{\mu_a} \otimes V_{\mu_b} \to V_{\mu_b} \otimes V_{\mu_a} \ .$$

We will denote the solitons by $A^{(a)}(\theta)$ in which θ is the rapidity. The scattering of two solitons of species a and b is then described by $S_{a,b}(\theta_{12})$, in which θ_{12} is their rapidity difference.

We then require some explicit crossing and fusion properties of the R-matrices. First, the Birman-Wenzl-Murakami algebra manipulations analogous to those of appendix B of [3] for the ${}_q c_n^{(1)}$ R-matrices give in the case of ${}_q a_{2n-1}^{(2)}$ R-matrices

$$\check{R}_{1,1}^{(TPG)cross}(-q^{2n} x_{1,1}^{-1}) = \frac{(q^2 - x_{1,1})(x_{11} + q^{2n})}{(1 - x_{1,1})(q^{2n} + x_{1,1} q^2)} \check{R}_{1,1}^{(TPG)}(x_{1,1}) \ .$$

Using (6.1) we can rewrite this equation in terms of θ and λ

$$c_{1,1}(i\pi - \theta) \check{R}_{1,1}^{cross}(x_{1,1}(i\pi - \theta)) = c_{1,1}(\theta) \check{R}_{1,1}(x_{1,1}(\theta)) \ ,$$

in which

$$c_{1,1}(\theta) = \sin(\pi(\mu - \omega)) \sin(\pi(\mu - n\omega + \frac{1}{2})) \ . \tag{6.4}$$

The general result is then

$$c_{a,b}(i\pi - \theta) \check{R}_{a,b}^{(TPG)cross}(x_{a,b}(i\pi - \theta)) = c_{a,b}(\theta) \check{R}_{a,b}^{(TPG)}(x_{a,b}(\theta))$$

in which

$$c_{a,b}(\theta) = \prod_{k=1}^{b} \sin(\pi(\mu - \frac{\omega}{2}(a - b + 2k))) \sin(\pi(\mu - \frac{\omega}{2}(2n - a - b + 2k) + \frac{1}{2})) \ .$$

This result will be combined with $k_{a,b}$ in the manner described in [3] in calculating the soliton-breather S-matrices.

The construction of the scalar factor $F_{a,b}(\mu)$ is very similar to the construction in [3] and will therefore only be described briefly here. $F_{a,b}(\mu)$ must ensure that the S-matrix satisfies unitarity

$$S_{a,b}(\theta) S_{b,a}(-\theta) = I_b \otimes I_a \tag{6.5}$$

and crossing symmetry

$$S_{a,b}(\theta) = [\sigma S_{b,a}(i\pi - \theta)]^{t_2} \sigma \ . \tag{6.6}$$

Considering the case $a = b = 1$ first and writing $F_{1,1}(\mu) = c_{1,1}(\theta) f_{1,1}(\mu)$, the equations (6.5) and (6.6) lead to the following two conditions on $f_{1,1}$:

$$
\begin{aligned}
f_{1,1}(-\mu + n\lambda) &= f_{1,1}(\mu) && (6.7)\\
f_{1,1}(\mu) f_{1,1}(-\mu) &= c_{1,1}^{-1}(\theta) c_{1,1}^{-1}(-\theta) \,. && (6.8)
\end{aligned}
$$

A solution to these equations is given by (see [3])

$$
f_{1,1}(\mu) = \prod_{j=1}^{\infty} \frac{f^{(1)}[\mu + 2n\lambda(j-1)] f^{(1)}[-\mu + 2n\lambda(j-\frac{1}{2})]}{f^{(1)}[\mu + 2n\lambda(j-\frac{1}{2})] f^{(1)}[-\mu + 2n\lambda j]} \tag{6.9}
$$

for any function $f^{(1)}(\mu)$ with $f^{(1)}(\mu) f^{(1)}(-\mu) = c_{1,1}^{-1}(\theta) c_{1,1}^{-1}(-\theta)$. Writing the right hand side of the equation (6.8) in terms of Gamma functions we can see that an appropriate starting function $f^{(1)}(\mu)$ is given by:

$$
f^{(1)}(\mu) = \frac{1}{\pi^2} \Gamma(\mu - \omega) \Gamma(\mu - n\omega + \frac{1}{2}) \Gamma(1 + \mu + \omega) \Gamma(\mu + n\omega + \frac{1}{2}) \,. \tag{6.10}
$$

Inserting this into (6.9) we obtain the overall scalar factor of $S_{1,1}$ in the following form

$$
\begin{aligned}
F_{1,1}(\mu) \quad = \quad & \prod_{j=1}^{\infty} \frac{\Gamma(\mu + 2n\lambda j - \omega)\Gamma(\mu + 2n\lambda j - (2n-1)\omega)}{\Gamma(-\mu + 2n\lambda j - \omega)\Gamma(-\mu + 2n\lambda j - (2n-1)\omega)} \\
& \times \frac{\Gamma(\mu + 2n\lambda j - n\omega + \frac{1}{2})\Gamma(\mu + 2n\lambda j - n\omega - \frac{1}{2})}{\Gamma(-\mu + 2n\lambda j - n\omega + \frac{1}{2})\Gamma(-\mu + 2n\lambda j - n\omega - \frac{1}{2})} \\
& \times \frac{\Gamma(-\mu + 2n\lambda j - (n+1)\omega - \frac{1}{2})\Gamma(-\mu + 2n\lambda j - (n-1)\omega + \frac{1}{2})}{\Gamma(\mu + 2n\lambda j - (n+1)\omega - \frac{1}{2})\Gamma(\mu + 2n\lambda j - (n-1)\omega + \frac{1}{2})} \\
& \times \frac{\Gamma(-\mu + 2n\lambda j - 2n\omega)\Gamma(-\mu + 2n\lambda j)}{\Gamma(\mu + 2n\lambda j - 2n\omega)\Gamma(\mu + 2n\lambda j)} \,.
\end{aligned} \tag{6.11}
$$

Starting with $S_{1,1}$ we examine the pole structure of the proposed S-matrix in order to obtain the poles at which two solitons of species a and b fuse together to a soliton of species $a+b$ (if $a + b \leq n$). We find that this process is possible if the rapidity difference of the incoming solitons is $\theta = i\frac{(a+b)\pi}{n}(1 - \frac{1}{2n\lambda})$. This corresponds to $\mu = \frac{a+b}{2}\omega$ and we can therefore write the general scalar factor $F_{a,b}(\mu)$ in the following form

$$
F_{a,b}(\mu) = \prod_{j=1}^{a} \prod_{k=1}^{b} F_{1,1}(\mu + \frac{\omega}{2}(2j + 2k - a - b - 2)) \,. \tag{6.12}
$$

These poles correspond to the R-matrix singularities due to horizontal links in the TPG; the vertical links give the crossed-channel poles.

From the knowledge of these fusing poles we are also able to deduce the exact quantum masses of the elementary solitons by using the usual mass formula

$$M_{a+b}^2 = M_a^2 + M_b^2 + 2M_aM_b\cosh(\mathrm{Im}\theta). \tag{6.13}$$

The quantum soliton masses are then

$$M_a = 4\sqrt{2}\frac{Chm}{\beta^2}\sin\left(\frac{a\pi}{h}\left(1-\frac{1}{h\lambda}\right)\right) \tag{6.14}$$

which could also be obtained from the classical soliton masses $M_a^{(class.)} = -4\sqrt{2}\frac{hm}{\beta^2}\sin\frac{a\pi}{h}$ by shifting the Coxeter number h (here $h=2n$) to the so-called quantum Coxeter number H:

$$h \to H = h + \frac{1}{\omega}\ .$$

Expanding (6.14) in terms of β^2 we obtain

$$M_a = 8\sqrt{2}\frac{Chm}{\beta^2}\sin\left(\frac{a\pi}{h}\right)[1-\beta^2\frac{a}{4h^2}\cot\left(\frac{a\pi}{h}\right)] + O(\beta^2)$$

which implies that (with an appropriate choice of the overall scale factor C) the mass correction ratios are those of the particles (l.h.s. of [6], table 5) and thus as suggested in the note to that paper and not as calculated there for the solitons (r.h.s. of table 5). We have not yet been able to resolve this discrepancy, and the semiclassical mass corrections can not therefore be regarded as understood.

6.2 Breather S-matrices

The so-called breathers are scalar bound states of two elementary solitons[7]. The rapidity difference at which two incoming solitons can fuse together to a breather can be determined from the spectral decomposition of the R-matrix (4.2). Since the breathers transform under the singlet representation the poles are contained in the prefactor of $\check{P}_0$, which can only appear in (4.2) if $a=b$. This means that only solitons of the same species can build scalar bound states. We will denote the scalar bound state of two solitons of species a as $B_p^{(a)}$, in which p is an excitation number.

The construction of the soliton-breather S-matrices and the breather-breather S-matrices uses the 'bootstrap' or 'fusion' procedure and is completely analogous to the construction in the case

[7]Scalar bound states necessarily have zero topological charge, but the opposite is not true, as many of the soliton particle multiplets contain particles of zero topological charge

of $d^{(2)}_{n+1}$ ATFT as done in [3]. However in the case of $b^{(1)}_n$ there are some differences regarding the nth breathers. Since we will identify the lowest breather states with the fundamental quantum particles of the $b^{(1)}_n$ theory in the next section, we expect the fusion of two solitons of type n into a breather to occur at a different pole than for all other breathers.

Usually, for algebras where all the R-matrices are known (as applied to the $d^{(2)}_{n+1}$ case [3]), the fusion poles in the scalar factor mimic the fusion properties of the R-matrix rather beautifully. Here, however, we can only calculate the R-matrices using the TPG for $a + b \leq n$. Beyond that, pending a general way of treating the higher R-matrices, we only have the information for the scalar factor. In particular our suggestion for the n-th breather pole remains somewhat speculative and can only be justified by the comparison with the real affine Toda S-matrices in section 4.3. We expect the nth breather to be roughly half as massive as might be expected from the naïve R-matrix considerations which led to (46) of [3]. This is not necessarily a problem, however; for example, it would be resolved if (the unknown) $R_{n,n}$ were to have singularities at both $\tilde{x} = -\tilde{q}^{2n}$ (from crossing symmetry) and $\tilde{x} = \tilde{q}^{2n}$ (expected from fusion).

We only list the results of the fusion calculations here (for details of the calculation see [20, 3]).

Two solitons of type a fuse to give a breather of type a at the poles:

$$\begin{aligned} \mu &= n\omega - p + \frac{1}{2} \qquad (\text{for } a = 1, 2, ..., n-1; \text{ and } p = 1, 2, ... \leq n\omega + \frac{1}{2}) \\ \mu &= n\omega - \frac{p}{2} + \frac{1}{2} \qquad (\text{for } a = n; \text{ and } p = 1, 2, ... \leq 2n\omega + 1) \end{aligned} \tag{6.15}$$

In order to write down all breather S-matrices in a compact form we define the blocks

$$\Big(y\Big) \equiv \frac{\sin(\frac{\pi}{2n\lambda}(\mu + x))}{\sin(\frac{\pi}{2n\lambda}(\mu - x))} \tag{6.16}$$

and

$$\Big\{y\Big\} \equiv \Big(y\Big)\Big(n\omega + \frac{1}{2} - y\Big)\Big(\omega + 1 - y\Big)\Big(n\omega - \omega - \frac{1}{2} + y\Big) \tag{6.17}$$

which have the following properties:

$$\begin{aligned} \Big\{n\omega + \frac{1}{2} - y\Big\} &= \Big\{y\Big\} \qquad (\text{crossing symmetry}) \\ \Big\{y \pm 2n\omega \pm 1\Big\} &= \Big\{y\Big\} \qquad (2\pi i \text{ periodicity}). \end{aligned}$$

The S-matrix elements for the scattering of breathers are the following:

Soliton-breather scattering $(a = 1, 2, ..., n;\ b = 1, 2, ..., n-1)$:

$$
\begin{aligned}
S_{A^{(a)}B_p^{(b)}}(\theta) &= \prod_{l=1}^{p}\prod_{k=1}^{b}\left\{\frac{\omega}{2}(2k-a-b+n)+l-\frac{p}{2}+\frac{1}{4}\right\}, \\
S_{A^{(a)}B_p^{(n)}}(\theta) &= \prod_{l=1}^{p}\prod_{k=1}^{a}\left(\frac{\omega}{2}(2k-2-a+2n)+\frac{l}{2}-\frac{p}{4}-\frac{1}{4}\right)\left(\frac{\omega}{2}(2k-a)+\frac{l}{2}-\frac{p}{4}\right).
\end{aligned}
\tag{6.18}
$$

Breather-breather scattering $(a, b = 1, 2, ..., n-1)$:

$$
\begin{aligned}
S_{B_r^{(a)}B_p^{(b)}}(\theta) &= \prod_{l=1}^{p}\prod_{k=1}^{b}\left\{\frac{\omega}{2}(2k-a-b+2n)+l-\frac{p+r}{2}+\frac{1}{2}\right\}\left\{\frac{\omega}{2}(2k-a-b)-l+\frac{p+r}{2}+1\right\}, \\
S_{B_r^{(a)}B_p^{(n)}}(\theta) &= \prod_{l=1}^{p}\prod_{k=1}^{a}\left\{\frac{\omega}{2}(2k+n-a)+\frac{l}{2}-\frac{r}{2}-\frac{p}{4}+\frac{1}{2}\right\}, \\
S_{B_r^{(n)}B_p^{(n)}}(\theta) &= \prod_{l=1}^{p}\prod_{k=1}^{n}\left\{\omega k+\frac{l}{2}-\frac{p+r}{4}+\frac{1}{2}\right\}.
\end{aligned}
\tag{6.19}
$$

6.3 Breather-particle identification

In this section we show that the S-matrix elements (6.19) for the lowest breathers ($p = 1$), i.e. the breathers with lowest mass, coincide with the S-matrices for the fundamental quantum particles as found for real coupling $b_n^{(1)}$ ATFT by Delius et al. in [18]. This identification of the lowest breathers with the fundamental particles has been demonstrated previously only for the cases of sine-Gordon theory [19], $a_2^{(1)}$ [20], $a_2^{(2)}$ [21] and $d_{n+1}^{(2)}$ ATFT [3]. The identification of the $b_n^{(1)}$ breathers with the $b_n^{(1)}$ particles shows in particular that there is no breather-particle Lie duality as suggested in [3], since in this latter case the $b_n^{(1)}$ breathers would have to be identified with the $c_n^{(1)}$ particles instead.

We want to compare the formulas (6.19) with the S-matrix for the real $b_n^{(1)}$ ATFT (see [18]):

$$
\begin{aligned}
S_{ab}^{(r)}(\theta) &= \prod_{k=1}^{b}\Big\{2k+a-b-1\Big\}_H\Big\{H-2k-a+b+1\Big\}_H \\
S_{an}^{(r)}(\theta) &= \prod_{k=1}^{a}\left\{\frac{1}{2}H+2k-a-1\right\}_H \\
S_{nn}^{(r)}(\theta) &= \frac{-1}{\left(\frac{1}{2}B\right)_H\left(H-\frac{1}{2}B\right)_H}\prod_{k=1}^{n-1}\frac{\Big(2k\Big)_H\Big(H-2k\Big)_H}{\Big(2k-B\Big)_H\Big(H-2k+B\Big)_H}
\end{aligned}
\tag{6.20}
$$

in which

$$\left\{y\right\}_H = \frac{(y-1)_H (y+1)_H}{(y-1+B)_H (y+1-B)_H}, \qquad (y)_H = \frac{\sin(\frac{\theta}{2i} + \frac{y\pi}{2H})}{\sin(\frac{\theta}{2i} - \frac{y\pi}{2H})}$$

and $H = h - \frac{1}{2}B$, $B = \frac{2\beta^2}{4\pi+\beta^2}$, $a, b = 1, 2, ..., n-1$. If we analytically continue $\beta \to i\beta$ we have to change

$$H \to 2n\frac{\lambda}{\omega}, \qquad B \to -\frac{2}{\omega},$$
$$\left(y\right)_H \quad \to \quad \left(\frac{\omega}{2}y\right) .$$

Doing this in the formulas (6.20) we obtain as expected

$$\begin{aligned} S^{(r)}_{ab}(\theta) &\to S_{B_1^{(a)} B_1^{(b)}}(\theta) \\ S^{(r)}_{an}(\theta) &\to S_{B_1^{(a)} B_1^{(n)}}(\theta) \\ S^{(r)}_{nn}(\theta) &\to S_{B_1^{(n)} B_1^{(n)}}(\theta) , \end{aligned}$$

and thus establish the identification of the $b_n^{(1)}$ lowest breathers with the $b_n^{(1)}$ quantum particles.

6.4 Particle spectrum and figures

Besides the breathers there are also bound states which non-zero topological charge present in the theory, which we will call excited solitons. These bound states correspond to the poles $\mu = a\omega - p$ (for $p = 0, 1, ... \le a\omega$, if $a \le \frac{n}{2}$) at which the S-matrix element $S_{a,a}(\theta)$ projects onto the representation space $V_{\mu_{2a}}$ (i.e. $S_{a,a} \sim \check{P}_{\mu_{2a}}$).

We therefore conjecture that $b_n^{(1)}$ quantum affine Toda field theory with imaginary coupling constant contains the following spectrum of solitons and bound states:

1) *fundamental solitons* $A^{(a)}$ $(a = 1, 2, .., n)$:
   masses $M_a = C 8\sqrt{2}\frac{2nm}{\beta^2} \sin(\frac{a\pi}{2n}(\frac{1}{2} - \frac{1}{2n\lambda}))$

2) *breathers* $B_p^{(a)}$ $(A^{(a)} - A^{(a)}$ bound states):
   masses $m_{B_p^{(a)}} = 2M_a \sin(\frac{p\pi}{2n\lambda})$ (for $a = 1, 2, ..., n-1$ and $p = 1, 2, ... \le n\omega + \frac{1}{2}$):
   and $m_{B_p^{(n)}} = 2M_n \sin(\frac{p\pi}{4n\lambda})$ (for $p = 1, 2, ... \le 2\omega + 1$):

3) *excited solitons* $A_p^{(2a)}$ $(A^{(a)} - A^{(a)}$ bound states) $(p = 0, 1, 2, ... \le a\omega)$
   masses $m_{A_p^{(2a)}} = 2M_a \cos(\frac{a\pi}{2n}(1 - \frac{1}{2n\lambda} - \frac{p}{a\lambda}))$.

We are confident of this conjecture, despite the fact that we can only check the pole structure of the R-matrices for $a + b \leq n$. There are a large number of poles which we do not expect to correspond to the fusion into bound states. As we saw in the $d_{n+1}^{(2)}$ case, it is not enough merely that simple poles, corresponding to R-matrix singularities, exist. Such poles can often be explained not as bootstrap poles but in terms of the existing spectrum using generalized Coleman-Thun methods, and is why odd-species excited solitons do not seem to occur. Our conjectures can only be verified by a large body of evidence matching poles to diagrams, and this will involve much additional work.

The following diagrams show three point vertices involving elementary solitons, breathers and excited solitons. We were not able to find any other three point couplings involving elementary solitons and breathers only. There are however other vertices involving excited solitons. Note in particular the differences to the list of three point couplings in $d_{n+1}^{(2)}$ ATFT [3]: the figures 1c and 1e reflect the slightly different role of the nth breather, which occurs at a different pole from all other breathers. Figure 1g is a new fusion process which does not exist in the $d_{n+1}^{(2)}$ case. This process is also consistent with the breather-particle identification, since the nth particle in real $b_n^{(1)}$ ATFT couples to all other particles [16]. A generalization of figure 1g to higher breathers ($p > 1$) does not seem to exist.

In the diagrams we have used the abbreviation $\tilde{\mu} = \mu(i\pi) = n\omega + \frac{1}{2}$. The imaginary angles correspond to the rapidity difference of the incoming particles. Time is meant to be moving upwards in all diagrams, but the processes obtained by turning any of the diagrams by 120 degrees are also allowed. In figure 1a and 1e a, b can take values in $1, 2, ..., n$ whereas in all other diagrams they take values in $1, 2, ..., n-1$.

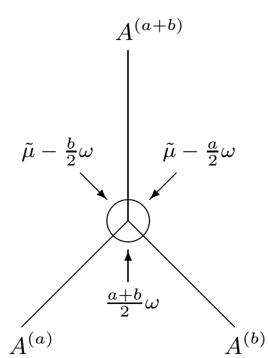

Figure 1a

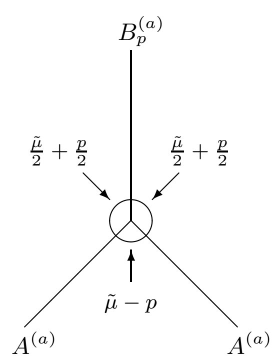

Figure 1b

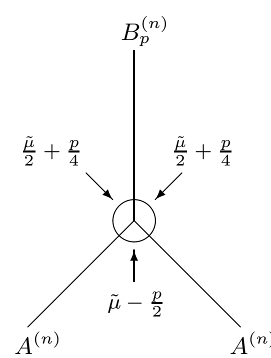

Figure 1c

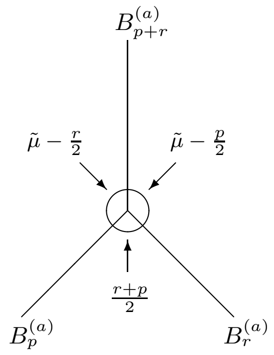

Figure 1d

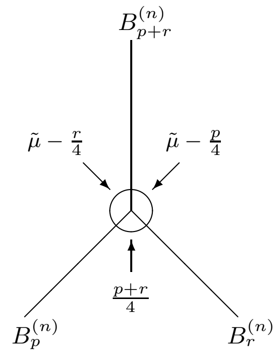

Figure 1e

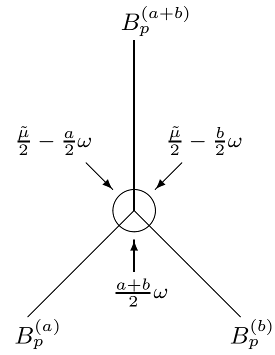

Figure 1f

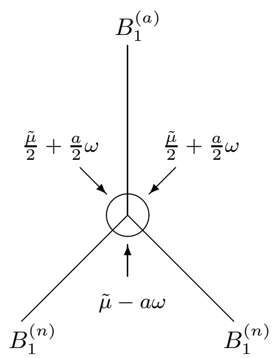

Figure 1g

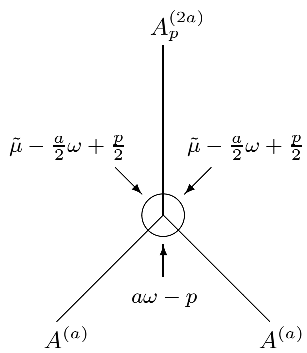

Figure 1h

6.5 Comments

As usual we have little positive evidence of any duality at this stage. In terms of the duality scheme introduced in [3], we make the following comments:

1. The $g^{(1)}$ and $g^{\vee(1)\vee}$ quantum soliton multiplets are both ${}_qg^{\vee}$ modules, but they will be different, e.g. the $d_{n+1}^{(2)}$ and $b_n^{(1)}$ soliton multiplets both form ${}_qc_n$ modules, but for $d_{n+1}^{(2)}$ these are irreducible, whereas for $b_n^{(1)}$ they are reducible.

   However, as noted earlier, the ${}_{q^2}c_n \subset {}_{q^2}c_n^{(1)}$ and ${}_qd_n \subset {}_qa_{2n-1}^{(2)}$ R-matrices used for constructing their S-matrices are related by $q^{2n+2} \mapsto -q^{2n}$. The basic (vector) representation has the same dimension in both cases, but the decompositions of its tensor products, and thus the dimensions of the higher multiplets, are different. Whether this transformation relates two physical theories is not yet clear.

2. The lowest b_n breathers are to be identified with $b_n^{(1)}$ ATFT particles, not $c_n^{(1)}$ particles as

was implied in [3]. There is therefore no breather-particle Lie duality. The particles exhibit a strong-weak affine duality [16], and we now expect the identification in each ATFT of the particles with the lowest breathers, which form a small part of a rich spectrum. The issue thus becomes how this affine duality fits into the overall picture.

3. Application of the $\beta \mapsto 4\pi/\beta$ transformation to higher breather S-matrices does not produce any simple effect.

There may be some significance in (4.6). (For example, consider g_0 and g to be c_n and a_{2n-1} respectively.) We obtain the $g_0^{\vee(1)}$ (*e.g.* $b_n^{(1)}$) ATFT by folding the $g'^{(1)}$ (*e.g.* $d_{n+1}^{(1)}$) ATFT, where $g'^{(1)}$ is the simply-laced algebra corresponding to $g'_0 = g_0^\vee$. Classically the solitons of the former are multi-solitons of the latter. But the $g_0^{\vee(1)}$ ATFT has a ${}_q g_0^{\vee(1)\vee}$ (*e.g.* ${}_q a_{2n-1}^{(2)}$) charge algebra, so that by (4.6) its soliton multiplets are multiplets of ${}_q g^{(k)}$ and therefore, as observed earlier, of ${}_q g^{(1)}$ (*e.g.* ${}_q a_{2n-1}^{(1)}$). Should they be identified with a subset of the single $g^{(1)}$ solitons? They have the same classical masses and quantum multiplets. One could not expect the quantum masses to be the same, since the corrections to the latter would now be calculated using only a subspace of the full $g^{(1)}$ theory. But there are also more fusings than would be expected with this identification: to give another example, in the ${}_q e_6^{(2)}$ case discussed above, the $e_6^{(1)}$ theory had $11 \to 1$ and $11 \to 3$ fusings, but the ${}_q e_6^{(2)}$ R-matrix and thus the $f_4^{(1)}$ soliton had in addition a $11 \to 2$ fusing. We leave this as a tantalising fact to be enlarged upon.

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References

[1] G.W. Delius, M.D. Gould and Y.-Z. Zhang, *Twisted quantum affine algebras and solutions to the Yang–Baxter equation*, preprint KCL–TH–95–8, q-alg/9508012

[2] G. Felder and A. LeClair, *Restricted quantum affine symmetry of perturbed minimal conformal models*, Int. J. Mod. Phys. **A7** (1992) 239; hep-th/9109019; D. Bernard and A. LeClair, *Quantum group symmetries and nonlocal currents in 2-D QFT,* Commun. Math. Phys. **142** (1991) 99.

[3] G.M. Gandenberger and N.J. MacKay, *Exact S-matrices for $d_{n+1}^{(2)}$ affine Toda solitons and their bound states,* preprint DAMTP-95-33, hep-th/9506169 (Nucl.Phys. B in press)

[4] N. Yu. Reshetikhin, *Quantised universal enveloping algebras, the Yang-Baxter equation and invariants of links, I*, LOMI preprint E-4-87.

[5] M. Jimbo, *Quantum R-matrix for the generalized Toda system*, Commun. Math. Phys. **102** (1986), 537

[6] N.J. MacKay and G.M.T. Watts, *Quantum mass corrections for affine Toda solitons*, Nucl. Phys. **B441** (1995), 277; `hep-th/9411169`

[7] V. Chari and A. Pressley, *Fundamental representations of Yangians and singularities of R-matrices*, J. für die reine und ang. Math. **417** (1991), 417

[8] *'Lie'*, produced by the Centre for Computer Algebra, Amsterdam.

[9] W.G. McKay and J. Patera, *Tables of dimensions, indices and branching rules for representations of simple Lie algebras*, Dekker (NY), 1981.

[10] N.J. MacKay, *Rational R-matrices in irreducible representations*, J. Phys. A: Math. Gen. **24** (1991), 4017

[11] R.B. Zhang, M.D. Gould and A.J. Bracken, *From representations of the braid group to solutions of the Yang-Baxter equation*, Nucl. Phys. **B354** (1991), 625

[12] G.W. Delius, M. Gould and Y.-Z. Zhang, *On the construction of trigonometric solutions of the Yang-Baxter equation*, Nucl. Phys. **B432** (1994), 377; `hep-th/9405030`

[13] M.-L. Ge, *New solutions of Yang-Baxter equations and quantum group structures*, Springer LNM 1510 (1990) 245, and references therein.

[14] N.J. MacKay, *The fusion of R-matrices using the Birman-Wenzl-Murakami algebra*, J. Math. Phys. **33** (1991), 1529

[15] S. Artz, L. Mezincescu and R. Nepomechie, *Analytic Bethe ansatz for quantum algebra invariant open spin chains*, preprint BONN-TH-95-09, UMTG-185, `hep-th/9504085`

[16] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, *Affine Toda field theory and exact S-matrices*, Nucl. Phys. **B338** (1990), 689; and *Multiple poles and other features of affine Toda field theory*, Nucl. Phys. **B356** (1991), 469

[17] E. Corrigan, *Recent developments in affine Toda quantum field theory*, Lectures given at CRM-CAP Summer School on Particles and Fields '94, Banff, Canada, 16-24 Aug 1994, preprint DTP-94/55, `hep-th/9412213`

[18] G.W. Delius, M.T. Grisaru and D. Zanon, *Exact S-matrices for non simply-laced affine Toda theories*, Nucl. Phys. **B382** (1992), 365

[19] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models*, Ann. Phys. (NY) **120** (1979), 253

[20] G.M. Gandenberger, *Exact S-matrices for bound states of $a_2^{(1)}$ affine Toda solitons*, Nucl.Phys. **B449** (1995), 375; `hep-th/9501136`

[21] F.A. Smirnov, *Exact S-matrices for $\phi_{1,2}$-perturbed minimal models of conformal field theory*, Int. J. Mod. Phys. **A6** No. 8 (1991), 1407

[22] T.J. Hollowood, *Quantizing $sl(N)$ solitons and the Hecke algebra*, Int. J. Mod. Phys. **A8** No.5 (1993), 947

Table 1: Subalgebras and multiplets

$g^{(k)}$	g	g_0	g_0'
$a_{2n}^{(2)}$	a_{2n}	b_n	c_n
$\imath$		1	2
$\tilde{V}$		$V_{2\lambda_1}$	V_1
Multiplets	$V_r\ ,\quad r = 1, .., n–1$	$V_r\ ,\quad r = 1, .., n–1$	$\oplus_{i=1}^{r} V_i\ ,\quad r = 1, .., n–1$
	V_n	$V_{2\lambda_n}$	$\oplus_{i=1}^{n} V_i$
$a_{2n-1}^{(2)}$	a_{2n-1}	c_n	d_n
$\imath$		2	1
$\tilde{V}$		V_2	$V_{2\lambda_1}$
Multiplets	$V_r\ ,\quad r = 1, .., n–2$	$\oplus_{i=0}^{[r/2]} V_{r-2i}\ ,\quad r = 1, .., n–2$	$V_r\ ,\quad r = 1, .., n–2$
	V_{n-1}	$\oplus_{i=0}^{[(n-1)/2]} V_{n-1-2i}$	$V_{\lambda_{n-1}+\lambda_n}$
	V_n	$\oplus_{i=0}^{[n/2]} V_{n-2i}$	$V_{2\lambda_{n-1}} \oplus V_{2\lambda_n}$
$d_{n+1}^{(2)}$	d_{n+1}	b_n	—
$\imath$		2	
$\tilde{V}$		V_1	
Multiplets	$V_r \oplus V_{r-2} \oplus ..\ ,\ r = 1, .., n–1$	$\oplus_{i=1}^{r} V_i\ ,\quad r = 1, .., n–1$	
	V_n	V_n	
$e_6^{(2)}$	e_6	f_4	c_4
$\imath$		2	2
$\tilde{V}$		V_4	V_4
Multiplets	V_1	$V_4 \oplus V_0$	V_2
	$V_2 \oplus V_0$	$V_1 \oplus V_4 \oplus V_0$	$V_4 \oplus V_{2\lambda_1} \oplus V_0$
	$V_3 \oplus V_6$	$V_3 \oplus V_1 \oplus 2V_4 \oplus V_0$	$V_{\lambda_1 \oplus \lambda_3} \oplus V_{2\lambda_1} \oplus V_2$
	$V_4 \oplus V_{\lambda_1+\lambda_6} \oplus 2V_2 \oplus V_0$	$\ldots$	$\ldots$
$d_4^{(3)}$	d_4	g_2	a_2
$\imath$		3	1
$\tilde{V}$		V_2	$V_{3\lambda_1}$
Multiplets	V_1	$V_2 \oplus V_0$	8
	$V_2 \oplus V_0$	$V_1 \oplus 2V_2 \oplus V_0$	$10 \oplus \overline{10} \oplus 8 \oplus 1$

In this table and throughout the paper we used [8] and [9] for Lie algebra information. The Dynkin diagrams and their numberings are given in table 1 of [6].

Table 2:

$${}_{q^2}c_n \subset {}_{q^2}c_n^{(1)}$$

$$\begin{tikzcd} V_{2\lambda_1}^+ \arrow[r, "2"] \arrow[d, "2n+2"] & V_2^- \\ V_0^- & \end{tikzcd}$$

$${}_{q}d_n \subset {}_{q}d_n^{(1)}$$

$$\begin{tikzcd} V_{2\lambda_1}^+ \arrow[r, "2"] & V_2^- \arrow[d, "2n-2"] \\ & V_0^+ \end{tikzcd}$$

$q^{2n-2} \mapsto -q^{2n}$

$q^{2n+2} \mapsto -q^{2n}$

$${}_{q^2}c_n \subset {}_{q}a_{2n-1}^{(2)}$$

$$\begin{tikzcd} V_{2\lambda_1}^+ \arrow[r, "2"] & V_2^- \arrow[d, "2n"] \\ & V_0^- \end{tikzcd}$$

$${}_{q}d_n \subset {}_{q}a_{2n-1}^{(2)}$$

$$\begin{tikzcd} V_{2\lambda_1}^+ \arrow[r, "2"] \arrow[d, "2n"] & V_2^- \\ V_0^+ & \end{tikzcd}$$

Relative parities of the different V are indicated by $V^{\pm}$.